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ON THE EXISTENCE OF THE SOLUTION OF AN OPTIMAL CONTROL PROBLEM FOR TIME-LAG SYSTEMS

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R. T. IANUSHEVSKII
(Moscow)

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The existence is investigated of the solution of an optimal control problem with a performance index specified as an integral quadratic form of coordinates and controls of a plant described by a system of linear differential equations with lagging argument. A connection is established between the given problem and optimal problems of a special kind, for which the solution existence conditions have been investigated in detail.

1. We consider the problem of minimizing the functional

$$I = \int_0^\infty (x^t(t) A^{(0)} x(t) + x^t(t) A^{(1)} u(t) + u^t(t) A^{(1)t} x(t) + u^t(t) A^{(2)} u(t))\,dt \tag{1.1}$$

relative to the plant's equations of motion

$$x^{\cdot}(t) = \sum_{i=0}^{l} A_i x(t-\tau_i) + \sum_{i=0}^{r} B_i u(t-\theta_i) \tag{1.2}$$

$$-\tau_l \leqslant t \leqslant 0, \quad x(t) = \varphi_x(t); \quad -\theta_r \leqslant t \leqslant 0, \quad u(t) = \varphi_u(t)$$

$$0 = \tau_0 < \tau_1 < \ldots < \tau_l, \quad 0 = \theta_0 < \theta_1 < \ldots < \theta_r$$

Here $x(t) = \{x_1, \ldots, x_m\}$ and $u(t) = \{u_1, \ldots, u_n\}$ are the state vector and the control vector, respectively $\varphi_x(t)$ and $\varphi_u(t)$ are initial functions for which a continuous solution of Eq. (1.2) exists τ_i and θ_j are the time lags in the coordinates of the state vector (l deviations of the argument) and of the controls (r deviations of the argument), respectively; A_i, B_j and $A^{(s)} = \| a_{gk}^{(s)} \|$ are constant matrices; the superscript t denotes transposition. Equations of the type given characterize a wide class of engineering processes, of economic systems, etc.

From the results in [1 — 6] it follows that if a solution of the problem given, as a problem of analytic design [1 — 6], exists, then it has the form

$$u(t) = Kx(t) + \int_{-\tau_l}^{0} K_1(\xi)\, x(t+\xi)\, d\xi + \int_{-\theta_r}^{0} K_2(\sigma)\, u(t+\sigma)\, d\sigma \tag{1.3}$$

$$K = -(A^{(2)})^{-1} (B_0^t W_0 + B_{01}^t(0) + A^{(1)t})$$

$$K_1(\xi) = -(A^{(2)})^{-1} (B_0^t B_{02}(\xi) + P_{03}(\xi, 0))$$

$$K_2(\sigma) = -(A^{(2)})^{-1} (B_0^t B_{01}(\sigma) + P_{01}(0, \sigma))$$

Here the matrices $W_0 = W_0^t$, $B_{0i}(\xi)$ and $P_{0k}(\xi, \sigma) = P_{0k}^t(\sigma, \xi)$ $(i = 1, 2;\ k = 1, 2, 3)$ satisfy the system of equations presented in [6].

Let us now consider the problem of optimizing a functional of form (1.1) relative the plant's equations of motion that are approximations of (1.2), as follows from [6]:

$$\dot{x}_{NN^*}(t) = A_0 x_{NN^*}(t) + \sum_{i=1}^{l} A_i z_{\tau_i}(t) + B_0 u_{NN^*}(t) + \sum_{i=1}^{r} B_i z^*_{\theta_i}(t) \tag{1.4}$$

$$\tau_l N^{-1} z_i^{\cdot}(t) + z_i(t) = z_{i-1}(t), \quad i = 1, \ldots, N$$

$$\theta_r (N^*)^{-1} z_i^{*\cdot}(t) + z_i^*(t) = z^*_{i-1}(t), \quad i = 1, \ldots, N^*$$

$$z_0(t) = x_{NN^*}(t), \quad z_0^*(t) = u_{NN^*}(t)$$

$$z_{\tau_i}(t) = z_{N\tau_i\tau_l^{-1}}(t), \quad z^*_{\theta_i}(t) = z^*_{N^*\theta_i\theta_r^{-1}}(t)$$

$$z_i(0) = \frac{N}{\tau_l} \int_{\varkappa_i}^{\varkappa_{i-1}} x(\vartheta)\, d\vartheta, \quad z_i^*(0) = \frac{N^*}{\theta_r} \int_{\chi_i}^{\chi_{i-1}} u(\vartheta)\, d\vartheta \tag{1.5}$$

$$\varkappa_i = -i\tau_l N^{-1}, \quad \chi_i = -i\theta_r (N^*)^{-1}$$

When minimizing functional (1.1) relative to (1.4) we use $x_{NN^*}(t)$, $u_{NN^*}(t)$ and I_{NN^*} instead of $x(t)$, $u(t)$ and I, respectively; $z_i(t)$ and $z_i^*(t)$ are vectors of dimension m and n, approximately characterizing the effect of the time lags in the coordinates of the state vector and controls.

Existence conditions for the solution of optimization problems of this class have been given, for instance, in [7 — 10]; the optimal solution has the form

$$u_{NN^*}(t) = K x_{NN^*}(t) + \frac{\tau_l}{N} \sum_{i=1}^{N} K_1 [1-i]\, z_i(t) + \tag{1.6}$$

$$\frac{\theta_r}{N^*} \sum_{i=1}^{N^*} K_2 [1-i]\, z_i^*(t)$$

Here $K_1[j]$ and $K_2[j]$ are matrices of dimensions $n \times m$ and $n \times n$. The optimal controls (1.3) and (1.6) are chosen from the class of admissible controls $u_g(t) \in L_2(0, \infty)$. When (1.3) and (1.6) are fulfilled Eqs. (1.2) and (1.4) have continuous solutions $x(t) \in L_2(0, \infty)$ and $x_{NN^*}(t) \in L_2(0, \infty)$ ($x(t)$ satisfies (1.2) almost everywhere). The following theorem establishes the relation between the existence conditions for the solutions of the optimal problems of minimizing the quadratic functional (1.1) relative to Eqs. (1.2) and (1.4), respectively.

T h e o r e m. Suppose that we can find numbers N_0 and N_0^* such that the optimal solution $u_{NN^*}(t)$, $x_{NN^*}(t)$ of problems of form (1.1), (1.4) — (1.6) exists (does not exist) for all $N \geqslant N_0$ and $N^* \geqslant N_0^*$. Then the optimal solution $u(t)$, $x(t)$ of control (1.1), (1.2) exists (does not exist), and in case the optimal solution exists we can find $N_0(\delta)$ and $N_0^*(\delta)$ such that

$$\int_0^\infty \| u(t) - u_{NN^*}(t) \|^2 dt \leqslant \delta, \quad \int_0^\infty \| x(t) - x_{NN^*}(t) \|^2 dt \leqslant \delta \tag{1.7}$$

$$| I - I_{NN^*} | \leqslant k_0 \delta, \quad k_0 = \max_{ij} | a_{ij}^{(s)} |$$

for all $N \geqslant N_0$ and $N^* \geqslant N_0^*$. Here the symbol $\| \cdot \|^2$ denotes the square of the Euclidean norm of the corresponding vector.

In what follows system (1.2), (1.3) and, respectively, the minimization problem (1.1), (1.2) are called original, while system (1.4)—(1.6) and the minimization problem

(1.1), (1.4) are called approximate.

We remark that the system (1.2), (1.3) of linear equations with lagging argument is treated here as a limit system relative to (1.4) — (1.6). We assume that Eq. (1.3) has been obtained formally on the basis of Bellman's functional equation and that the control systems satisfied by the matrices W_0, B_{0i} (ξ) and $P_{0k}(\xi, \sigma)$ ($i = 1,2$; $k = 1, 2, 3$) is approximated by the difference scheme following from the algebraic Riccati equation characterizing the approximate optimal problems (1.1), (1.4) (for example, see [6,10]). while K_1 $[1 - i]$ and K_2 $[1 - j]$ are first-order approximations of matrices K_1 (ξ) and K_2 (σ), respectively, in the nodes $-i\tau_l N^{-1}$ and $j\theta_r (N^*)^{-1}$, i.e., $-i\,\tau_l N^{-1} \leqslant \xi \leqslant -(i-1)\,\tau_l N^{-1}$ and $-j\theta_r (N^*)^{-1} \leqslant \sigma \leqslant -(j-1)\,\theta_r (N^*)^{-1}$, $K_1(\xi) = K_1[i] + o(N^{-1})$ and $K_2(\sigma) = K_2[j] + o(N^{*-1})$.

L e m m a 1. Let $u(t)$, $x(t)$ and $u_{NN^*}(t)$, $x_{NN^*}(t)$ be asymptotically stable solutions of the original problem (1.2), (1.3) and of the approximate problem (1.4), (1.6), respectively, and let I and I_{NN^*} be the values of functional (1.1) under the values indicated. Then for every arbitrarily small $\delta > 0$ there exists numbers N_0 (δ) and N_0^* (δ) for which inequalities (1.7) are valid.

We remark that analogously to [11 — 14] we can establish the proximity of the solutions of the original system (1.2) of equations with lagging argument and of the approximate systems (1.4) with $N \geqslant N_0$ and $N^* \geqslant N_0^*$ for an arbitrary finite interval $[0, T_0]$, i.e., for any arbitrarily small δ_1 and for an arbitrary bounded domain of initial conditions of system (1.2), as well as for a bounded norm of $u(t) \in L_2(0, T_0)$, we can find $N_0(\delta_1)$ and $N_0^*(\delta_1)$ such that $\| x(t) - x_{NN^*}(t) \|_{L_2(0,T_0)} \leqslant \delta_1$.

Expressions permitting a direct setting up of trajectories $x(t)$ and $x_{NN^*}(t)$ follow from (1.3) and (1.6) under a substitution of the corresponding values of the coordinates of the original and the approximate plants, determined by solving Eqs. (1.2) and (1.4) relative to $x(t)$ and $x_{NN^*}(t)$. The difference $e_u(t) = u(t) - u_{NN^*}(t)$ is estimated successively on the two intervals $[0, T_0]$ and $[T_0, \infty)$. By virtue of the asymptotic stability of the systems the integral square estimate of $e_u(t)$ on the interval $[T_0, \infty)$ can be made as small as desired by means of choosing T_0. Let us estimate the solution $e_u(t)$ on the interval $[0, T_0]$ by investigating the equation for $e_u(t)$ on the interval $[-\theta, T_0]$, $\theta = \max\{\theta_r, \tau_l\}$, as a Volterra intergral equation of the second kind. Because of the square summability of this equation's kernel and because its free term tends to zero uniformly in $t < T_0$ as N and N^* tend to infinity [12, 15], the smallness of $\| e_u(t) \|_{L_2(0,T_0)}$ and, consequently, the fulfilment of the first of conditions (1.7) follow from the properties of the solution of the Volterra equation. The estimate for the difference $e_x(t) = x(t) - x_{NN^*}(t)$ is derived analogously. The resulting last inequality in (1.7) follows from the preceding two.

L e m m a 2. If system (1.2), (1.3) is asymptotically stable, for the existence of optimal control (1.3) it is necessary and sufficient that the matrices $W_0 = W_0^t$, $B_{0i}(\xi)$, $i = 1, 2$, and $P_{0k}(\xi, \sigma) = P_{0k}^t(\sigma, \xi)$, $k = 1, 2, 3$, satisfying a system of Riccati equations exist [6].

The proof is analogous to that in [1, 2].

2. The proof of the theorem stated in Sect. 1 is based on the investigation of the

asymptotic stability (instability) of system (1.2), (1.3). We reckon that the case of instability of the system encompasses the case of nonasymptotic stability. In the case of asymptotic stability the proximity (in the sense of (1.7)) of the solutions of equation systems (1.2), (1.3) and (1.4), (1.6) follows from Lemma 1. If we allow for the fact that $u(t)$ has been written in form (1.3) on the basis of Bellman's functional equation, then from Lemma 2 it follows that the given $u(t)$ is the optimal control.

In the case of instability of system (1.2), (1.3) functional (1.1) with $u(t)$ in form (1.3) is unbounded, i.e., the optimal solution is absent. Indeed, by virtue of the instability of system (1.2), (1.3) the inequality $\|x(t_k)\|^2 \geqslant 2\delta_0 > 0$ is fulfilled for a denumerable set of t_k. Using (1.2), (1.3) and the expression $x(t_k+\varepsilon) - x(t_k) \approx \varepsilon x^{\cdot}(t_k)$, we select the size of the interval $(t_k - \varepsilon_k, t_k + \varepsilon_k)$ on which $\|x(t)\|^2 \geqslant \delta_0$ with the aid of the relation

$$0 < \delta_2 = \varepsilon_k^2 \|x^{\cdot}(t_k)\|^2 \leqslant M\varepsilon_k^2 \max_{t \leqslant t_k} \|x(t)\|^2 \leqslant \delta_0, \quad M > 0$$

If the derivatives $x^{\cdot}(t_k)$ are bounded, then $\varepsilon_k \leqslant \varepsilon > 0$ and, consequently, the integral

$$I > \sum_k 2\varepsilon_k \delta_0$$

diverges. If $\lim_{k\to k_0} \|x(t_k)\|^2 = \infty$ for some of the points of subsequence t_{k_0}, then according to the relation cited $\lim_{k\to k_0} \varepsilon_k = 0$, whereas, $\lim_{k\to k_0} M\varepsilon_k^2 \|x(t_k)\|^2 \geqslant \delta_2$. Then the integral

$$I \geqslant \lim_{k\to k_0} \int_{t_k}^{t_k+\varepsilon_k} \|x(t)\|^2 dt = \lim_{k\to k_0} \|x(t_k)\|^2 \varepsilon_k \geqslant \lim_{k\to k_0} \frac{\delta_2}{M\varepsilon_k}$$

is unbounded.

System (1.2), (1.3) is investigated by comparison with the approximate systems (1.4), (1.6), and the instability of approximate system indicates the absence of an optimal solution to the corresponding approximate problem. Let us set up the trajectories $x_0(t) = \{x_{NN^*}(t), z_1(t), \ldots, z^*_{N^*}(t)\}$ of the optimal approximate system and $x(t)$ of system (1.2), (1.3). From (1.2) — (1.6) follow the inequalities

$$\|x(t) - x_{NN^*}(t)\| \leqslant \sum_{i=0}^{l} \|A_i\| \int_{t_1}^{t} \|x(\xi - \tau_i) - z_{\tau_i}(\xi)\| d\xi + \tag{2.1}$$

$$\sum_{i=0}^{r} \|B_i\| \int_{t_1}^{t} \|u(\sigma - \theta_i) - z_{\theta_i}^*(\sigma)\| d\sigma$$

$$S_x(t) \leqslant G_x(t) + \|x_0(t)\|, \quad S_u(t) \leqslant G_u(t) + \|u_{NN^*}(t)\|$$

$$\|u(t - \theta_i) - z^*_{\theta_i}(t)\| \leqslant G_u(t) + 2n\theta_i\gamma_u(t) / \sqrt{N^*} \tag{2.2}$$

$$\|x(t - \tau_i) - z_{\tau_i}(t)\| \leqslant G_x(t) + 2m\tau_i\gamma_x(t) / \sqrt{N}$$

Here

$$S_y(t) = \max_{\xi} \|y(\xi)\|, \quad t_0 \leqslant \xi \leqslant t; \quad \|y(t)\| = \max_i |y_i(t)|$$

$$y(t) = \{y_1(t), \ldots, y_k(t)\}; \quad G_x(t) = \max_{\xi} \|x(\xi) - x_{NN^*}(\xi)\|$$

$$G_u(t) = \max_{\xi} \|u(\xi) - u_{NN^*}(\xi)\|, \quad t_0 \leqslant \xi \leqslant t, \quad t_1 \geqslant t_0$$

Inequalities (2.2) have been written under the assumptions $\xi \leqslant t$, $\| x_{NN*}(\xi) \| \leqslant \gamma_x(t)$ and $\| u_{NN*}(\xi) \| \leqslant \gamma_u(t)$, and from Eq. (1.6) it follows that if $\| x_{NN*}(\xi) \| \leqslant \gamma_x(t)$, then a constant M_1 exists for which the relation

$$\gamma_u(t) \leqslant M_1 \gamma_x(t)$$

is fulfilled. Here and further the constants $M_i > 0$ in the estimates of the inequalities are used without being determined in detail.

According to Eq. (1.2) we have

$$t \geqslant t_0 + \theta, \ \| x^{\cdot}(t) \| \leqslant M_2 S_x(t) + M_3 S_u(t) \leqslant M_2 G_x(t) + M_3 G_u(t) + M_4 (\| x(t) \| + \| u(t) \|), \ \theta = \max(\tau_l, \theta_r) \tag{2.3}$$

Having substituted the appropriate quantities into expression (2.1), we present the latter as

$$G_x(t) \leqslant M_5 \int_{t_1}^{t} G_x(\xi)\,d\xi + M_6 \int_{t_1}^{t} G_u(\xi)\,d\xi + \left(\frac{M_7}{\sqrt{N}} + \frac{M_8}{\sqrt{N^*}}\right) \int_{t_1}^{t} (\| x(\xi) \| + \| u(\xi) \|)\,d\xi \tag{2.4}$$

From expressions (1.3) and (1.6) follows

$$\| u(t) - u_{NN*}(t) \| \leqslant \| K(x(t) - x_{NN*}(t)) \| + \| Q_1 \| + \| Q_2 \| + o(N_-^{-1}) \tag{2.5}$$

$$Q_1 = \frac{\tau_l}{N} \sum_{i=1}^{N} K_1 [1 - i] \left(\frac{N}{\tau_l} \int_{\varkappa_i}^{\varkappa_{i-1}} x(t + \xi)\,d\xi - z_i(t) \right)$$

$$Q_2 = \frac{\theta_r}{N^*} \sum_{i=1}^{N^*} K_2 [1 - i] \left(\frac{N^*}{\theta_r} \int_{\chi_i}^{\chi_{i-1}} u(t + \sigma)\,d\sigma - z_i^*(t) \right)$$

$$N_- = \min(N, N^*)$$

Using the relations

$$\left\| \frac{N}{\tau_l} \int_{\varkappa_i}^{\varkappa_{i-1}} (x(t + \xi) - x_{NN*}(t + \xi))\,d\xi + \frac{N}{\tau_l} \int_{\varkappa_i}^{\varkappa_{i-1}} x_{NN*}(t + \xi)\,d\xi - z_i(t) \right\| \leqslant \frac{N}{\tau_l} \int_{t+\varkappa_i}^{t+\varkappa_{i-1}} G_x(\xi)\,d\xi + \frac{2m\alpha_i}{\sqrt{N}} \gamma_x(t), \quad \frac{(i-1)\tau_l}{N} \leqslant \alpha_i \leqslant \frac{i\tau_l}{N}$$

we reduce expression (2.5) to

$$G_u(t) \leqslant M_9 G_x(t) + M_{10} \int_{t_1}^{t} G_u(\xi)\,d\xi + o(N_-^{-1}) \tag{2.6}$$

From inequalities (2.4) and (2.5) and the lemma in [11] it follows that when $\| x(t) \| + \| u(t) \| \leqslant \delta$, $t_1 \leqslant t \leqslant t_1 + T$,

$$G_{xu}(t) = G_x(t) + G_u(t) \leqslant (M_{12}N^{-1/2} + M_{13}N^{*-1/2})(J_1 + M_{14}J_2) + o(N_-^{-1})$$

$$J_1 = \int_{t_1}^{t} (\| x(\xi) \| + \| u(\xi) \|)\, d\xi, \quad J_2 = \int_{t_1}^{t}\int_{t_1}^{\xi} (\| x(\sigma) \| + \| u(\sigma) \|) \times \exp[(M_{12}N^{-1/2} + M_{13}N^{*-1/2})(t-\xi)]\, d\sigma\, d\xi$$

Taking into account that the solution of a system of linear differential equations with constant coefficients, describing a approximate optimal system, is majorized by a damped exponential function, we have

$$\| x_{NN^*}(t) \| + \| u_{NN^*}(t) \| \leqslant k_1\delta_1 e^{-\alpha(t-t_0)}, \quad k_1 > 1, \quad \alpha > 0$$
$$\| x_0(t_0) \| \leqslant \delta_1$$

Let the initial conditions of the system (1.2) and (1.3) are such that

$$t_0 \leqslant t \leqslant t_0 + \theta, \quad S_x(t) + S_u(t) \leqslant {}^1/_2\, \delta / k_1$$

Let us determine the initial conditions $x_0(t)$ at instant $t_0 + \theta$ from the corresponding values of $x(t)$ and $u(t)$ according to (1.5). Let $t_0 + T = \alpha^{-1}\ln(4k_1)$ and let N_- be so large that

$$t_0 + \theta \leqslant t \leqslant t_0 + \theta + T, \quad M_0(T) / \sqrt{N_-} \leqslant {}^1/_4$$

Then the inequalities

$$t_0 + \theta \leqslant t \leqslant t_0 + \theta + T, \quad \| x(t) \| + \| u(t) \| \leqslant G_{xu}(t) + \| x_{NN^*}(t) \| + \| u_{NN^*}(t) \| < \delta$$
$$t_0 + T \leqslant t \leqslant t_0 + \theta + T, \quad \| x(t) \| + \| u(t) \| \leqslant \delta / 4$$

are filfilled to within quantities of the order of smallness $o(N_-^{-1})$. Now assuming $t_0 + \theta + T$ as the initial instant t_1, we determine the initial values $x_0(t)$ from the corresponding values of $x(t)$ and $u(t)$. Analogously we obtain

$$t_0 + \theta + T \leqslant t \leqslant t_0 + \theta + 2T, \quad \| x(t) \| + \| u(t) \| \leqslant \delta / 4 + G_{xu}(t) < \delta / 2$$
$$t_0 + 2T \leqslant t \leqslant t_0 + \theta + 2T, \quad \| x(t) \| + \| u(t) \| \leqslant \delta / 8$$

Repeating for T more steps, we find

$$t_0 + kT \leqslant t \leqslant t_0 + \theta + kT, \quad \| u(t) \| + \| x(t) \| \leqslant 2^{-k-1}\delta$$
$$k = 1, 2, \ldots$$

Thus, for an arbitrary bounded domain of initial conditions of system (1.2), (1.3) we can find a time interval kT during which the trajectories of the coordinates and of the controls of the system belong to an arbitrarily small neighborhood of the origin (of the order $o(N_-^{-1})$), which attests to the asymptotic stability of the given system.

An insignificant change in the proof scheme presented (see [11], for instance) shows that if system (1.2), (1.3) is asymptotically stable, then we can find N_0 and N_0^* such that the approximate systems (1.4), (1.6) are asymptotically stable when $N \geqslant N_0$ and $N^* \geqslant N_0^*$. Finally, the instability of system (1.2), (1.3) when the approximate systems (1.4), (1.6) are unstable can be proved by contradiction.

In conclusion we mention that the results established above serve as a foundation for the possibility of investigating the considered class of optimal control problems for plants with time lags in the control and in the coordinates on the basis of the investigation of optimal control problems for linear dynamic plants without time lags, where the special form of the matrices of these plants' dynamics equations permits us to confine ourselves to finite, frequently small, values of N and N^*. This question is touched upon in more detail in [6] wherein the calculations are carried out for a number of concrete time-lag control systems. Also considered in [6] is a suboptimal control problem when plant (1.2) is controlled by regulator (1.6) whose parameters are determined by solving an approximate optimal problem. We take note also of [16] in which a method is considered for investigating a class of optimal control problems for distributed-parameter plants on the basis of an investigation of a sequence of finite-dimensional optimal problems.

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